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EDGE-COLORED COMPLETE GRAPHS WITH PRECISELY COLORED SUBGRAPHS

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Dedicated to Paul Erdős on his seventieth birthday

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Let $f(s, t; k)$ be the largest value of m such that it is possible to k-color the edges of the complete graph K_m so that every $K_s \subseteq K_m$ has exactly t colors occuring on its edges. The main object of this paper is to describe the behavior of the function $f(s, t; k)$, usually thinking of s and t fixed, and letting k become large.

1. Introduction

A classical question arising in Ramsey theory is the determination of $r(s; k)$, defined to be the *least* integer r such that in any partition of the edges of the complete graph K_r on r vertices into k classes, some class must contain *all* the edges of some complete subgraph K_s on s vertices. (For terminology in graph theory, see [1]; for general information on Ramsey theory, see [8] or [9]; for the current state of knowledge on the values of $r(s; k)$, see [3]). Stated another way, if $m<r(s; k)$ then there is a partition of the edges of K_m into k classes such that every complete subgraph $K_s \subseteq K_m$ has edges in at least two of the classes. We will usually use the alternative "chromatic" terminology in which the preceding result can be stated as follows: If $m<r(s; k)$ then there is a k-coloring of $E(K_m)$ such that no complete subgraph $K_s \subseteq K_m$ is monochromatic.

In this paper we will study this lack of "monochromaticity" in a more quantitative manner. Our investigations were initially stimulated by the following attractive question of T. A. Brown [2]: What is the largest number $f(k)$ of vertices a complete graph can have such that it is possible to k-color its edges so that every triangle $(=K_3)$ has edges of *exactly* two colors?

It follows at once that $f(k)$ exists and by Ramsey's Theorem, in fact, satisfies

$$f(k) < r(3; k).$$

In Theorem 1, we determine $f(k)$ exactly.

More generally, one can ask for the value of $f(s, t; k)$, defined to be the largest value of m such that it is possible to color the edges of K_m so that every $K_s \subseteq K_m$

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has *exactly* t different colors occurring on its edges. The main object of this paper is to describe the behavior of $f(s, t; k)$ as function of its arguments s, t and k, usually thinking of s and t fixed, and letting k become large.

We should point out here that a number of earlier papers of Erdős and others have dealt with related questions in which it is required, for example, that *at least* t colors appear in each (or some) K_s. A particularly nice result of this type, due to Erdős and Szemerédi, is the following:

Theorem [7]: *There exists a constant $c>0$ so that if $E(K_n)$ is k-colored then there is an $s \geqq \frac{ck \log n}{\log k}$ and a $K_s \subseteq K_n$ which spans only $k-1$ colors in its edge set.*

More results of this type can be found in [4], [6], [7].

2. Preliminaries

A k-coloring of the edge set $E=E(G)$ of a graph G will be thought of as a map χ: $E \to [k] := \{1, 2, ..., k\}$. If $X \subseteq E$, $C(X)$ will denote the set of colors occurring in X, and $c(X)$ will denote the cardinality of $C(X)$. That is,

$$C(X) = \{\chi(x) \colon x \in X\}, \quad c(X) = |C(X)|.$$

If $A, B \subseteq V = V(G)$, the set of vertices of G, then $[A, B]$ will denote the set of edges of G of the form $\{a, b\}$, $a \in A$, $b \in B$.

As mentioned earlier, for $s, t, k \in \mathbf{Z}^+$, the set of positive integers, define $f(s, t; k)$ by

$$f(s, t; k) = \max \{m \colon \text{ for some } \chi \colon E(K_m) \to [k],\ c(K_s) = t \text{ for every } K_s \subseteq K_m\}$$

where the abuse of notation with $c(K_s)$ has the obvious meaning. Of course, $f(s, t; k)$ is not well-defined for all values of s, t and k. In particular, we will always assume: $2 \leqq t \leqq k$, $t \leqq \binom{s}{2}$.

Quite relevant to our discussion will be the so-called *canonical* partition theorem of Erdős and Rado (see [5] or [9]). A particular consequence of it is the following result:

Theorem [5]: *For all s there exists an $r^*(s)$ such that if $m \geqq r^*(s)$ then in any coloring χ: $E(K_m) \to \mathbf{Z}^+$ there is a subgraph $K_s \subseteq K_m$ with $c(K_s) = 1$, $s-1$ or $\binom{s}{2}$.*

We have already ruled out consideration of the case $t=1$ since all edges must have the same color when $s \geqq 3$. On the other hand, if $t = \binom{s}{2}$ then $f(s, t; k)$ is not particularly interesting since we now require any $K_s \subseteq K_m$ to have all its edges with distinct colors. Thus, if $s \geqq 4$ then *all* edges of K_m itself must have distinct colors and $\binom{m}{2} \leqq k$. For the case $s=3$, it is easy to see that

$$f(3, 3; k) = \begin{cases} k+1 & \text{if } k \text{ is odd} \\ k & \text{if } k \text{ is even} \end{cases}$$

Since by the theorem of Erdős and Rado, $f(s, t; k)$ is *bounded* as $k \to \infty$ if $t \neq 1$, $s-1$ or $\binom{s}{2}$, and we have dispensed with the cases $t=1$ and $t=\binom{s}{2}$ then the only case of interest left is that of $t=s-1$. To simplify notation we define

$$f(s; k) = f(s, s-1; k)$$

$$f(k) = f(3; k)$$

We discuss these functions in the remainder of the paper.

3. The case $s=3$

Theorem 1.

$$f(k) = \begin{cases} 5^{k/2} & \textit{if} \quad k \textit{ is even} \\ 2 \cdot 5^{(k-1)/2} & \textit{if} \quad k \textit{ is odd} \end{cases}$$

Proof. First, we claim that

(1) $$f(k+2) \geqq 5f(k).$$

To see this, let $\chi\colon E(K_{f(k)}) \to [k]$ be a k-coloring of $E(K_{f(k)})$ in which every K_3 has exactly two colors. Consider the graph $K_{5f(k)}$ with the vertex set denoted by

$$\{x_i(j)\colon 1 \leqq i \leqq f(k), \quad 1 \leqq j \leqq 5\}.$$

Define a $(k+2)$-coloring χ^* of $E(K_{5f(k)})$ by:

$$\chi^*(x_i(j), x_{i'}(j')) = \begin{cases} \chi(x_i, x_{i'}) & \text{if} \quad j = j' \\ k+1 & \text{if} \quad j-j' \equiv \pm 1 \pmod 5 \\ k+2 & \text{if} \quad j-j' \equiv \pm 2 \pmod 5 \end{cases}$$

where $\chi(x_i, x_{i'})$ denotes the color assigned by χ to the edge $\{x_i, x_{i'}\}$ in $K_{f(k)}$. It is easy to check that χ^* is a $(k+2)$-coloring of $K_{5f(k)}$ (actually, its edges) in which every triangle has exactly two colors, and (1) is proved.

Note that $f(1)=2$ and $f(2)=5$. Also, a similar (but simpler) construction shows that $f(k+1) \geqq 2f(k)$ and consequently.

(2) $$f(k+d) \geqq 2^d f(k), \quad d \geqq 0.$$

It suffices to prove

(1′) $$f(k+2) \leqq 5f(k),$$

since this, together with the stated values for $f(1)$ and $f(2)$, implies the theorem.

Suppose $\chi\colon E(K_n) \to [k]$ is such that every triangle has exactly two colors, where $k \geqq 3$. It will be enough to show that

(3) $$n \leqq 5f(k-2).$$

Choose an arbitrary fixed vertex $x_0 \in V(K_n) = V$. For $i \in [k]$, define

$$N_i := \{y \in V: \chi(x_0, y) = i\}, \quad n_i := |N_i|, \quad c_i := C(N_i), \quad c_i := |C_i|,$$

Obviously $i \notin C_i$ since every K_3 is 2-colored.

Fact 1. *Suppose $i \in C_j$ and $j \in C_i$. Then N_i has a spanning complete bipartite subgraph in color j and N_j has a spanning complete bipartite subgraph in color i.*

Proof of Fact 1. First, observe that all edges in $[N_i, N_j]$ must have colors i or j. Suppose $\{\alpha, \beta\}$ is an edge in N_i with color j and $\{\alpha', \beta'\}$ is an edge in N_j with color i. Define

$$A := \{x \in N_j: \ \{\alpha, x\} \ \text{ has color } \ i\}, \quad B := \{y \in N_j: \ \{\alpha, y\} \ \text{ has color } \ j\}.$$

Thus, $A \cup B = N_j$.

(i) If $A \neq \emptyset$, $B \neq \emptyset$ then $u \in A$, $v \in B$ implies $\{u, v\}$ has color i since $K_3(\alpha, u, v)$, the triangle determined by α, u and v, has only color i and j, and $j \notin C_j$.

(ii) If $B = \emptyset$ then we reach a contradiction since $K_3(\alpha, \alpha', \beta')$ has only one color.

(iii) If $A = \emptyset$ then $x \in N_j$ implies $\{\beta, x\}$ has color i. But now $K_3(\beta, \alpha', \beta')$ has only color i edges which is impossible.

Hence, we are forced into the conclusion that (i) must hold, i.e., N_j can be written as $N_j = A \cup B$ with $A \neq \emptyset$, $B \neq \emptyset$ and all edges in $[A, B]$ have color i. By symmetry a similar conclusion applies to N_j and the Fact is proved. ∎

We will call such a monochromatic spanning complete bipartite subgraph in N_j an MSCBS. Note that a set $X \subseteq V$ can have at most one MSCBS.

Relabel the colors if necessary so that $c_1 \geqq c_i$ for all $i \in [k]$. There are several cases.

Case 1. N_1 does not contain an MSCBS. Thus, by Fact 1, $i \in C_1$ implies $1 \notin C_i$. Therefore,

$$n \leqq |N_1| + \Big|\Big\{\bigcup_{i \in C_1} N_i \cup \{x_0\}\Big\}\Big| + \Big|\Big\{\bigcup_{\substack{j \notin C_1 \\ j \neq 1}} N_j\Big\}\Big| \leqq f(c_1) + f(k-1) + \sum_{\substack{j \notin C_1 \\ j \neq 1}} f(c_j) \leqq$$

$$\leqq f(c_1) + f(k-1) + (k-1-c_1) f(c_1) = (k-c_1) f(c_1) + f(k-1) \leqq$$

$$\leqq \big((k-c_1) 2^{-(k-1-c_1)} + 1\big) f(k-1), \quad \text{by} \quad (2),$$

since $c_1 \leqq k-1$.

But

$$a \cdot 2^{1-a} \leqq 1, \quad a = 0, 1, 2, \ldots$$

and consequently $n \leqq 2f(k-1)$ which is actually stronger than (3).

Case 2. N_1 contains an MSCBS in color 2 but N_2 does not contain an MSCBS in color 1. As before, by Fact 1, $1 \notin C_2$. Also, $j \in C_1$, $j \neq 2$, implies $1 \notin C_j$. Thus $1 \notin \bigcup_{j \in C_1} C_j$. Thus, we can argue as before that

$$n \leqq |N_1| + \Big|\Big\{\bigcup_{j \in C_1} N_j \cup \{x_0\}\Big\}\Big| + \Big|\Big\{\bigcup_{\substack{j \notin C_1 \\ j \neq 1}} N_j\Big\}\Big| \leqq$$

$$\leqq f(c_1) + f(k-1) + (k-1-c_1) f(c_1) \leqq 2f(k-1).$$

Case 3. N_1 contains an MSCBS in color 2 and N_2 contains an MSCBS in color 1. Let $N_1=A\cup B$ and $N_2=A'\cup B'$ be the bipartitions induced by the hypothesized MSCBS's. If there are $a\in A$, $a'\in A'$, $b'\in B'$ with $\chi(a,a')=\chi(a,b')=2$ then for any $b\in B$, $\chi(b,a')=\chi(b,b')=1$ which is impossible. By this argument, we easily conclude that all edges $\{a,a'\}$, $a\in A$, $a'\in A'$ and $\{b,b'\}$, $b\in B$, $b'\in B'$ have the *same* color, say 1, and all other edges $\{a,b'\}$, $\{a',b\}$ have the other color, in this case, 2. We next *claim* that if $j\in C_1\cup C_2$, $j\neq 1,2$ then $1\notin C_j$, $2\notin C_j$. Suppose the contrary, say $j\in C_1$, $2\in C_j$. Thus, all edges in $[N_1, N_j]$ must have color 1. (They must have colors 1 or j and if any one has color j then because of the MSCBS in N_1, all must have color j, which is a contradiction). Similarly, all edges in $[N_2, N_j]$ must have color j. But now $K_3(a,b',x)$, $x\in N_j$, spans 3 colors, which is impossible.

A similar argument applies if $j\in C_2$, $1\in C_j$. Of course, we cannot have $j\in C_1$ and $1\in C_j$ since by Fact 1, this implies N_1 has an MSCBS in color $j\neq 2$ (and similarly, we cannot have $j\in C_2$ and $2\in C_j$). This proves the claim.

Thus, we have, in this case

$$\begin{aligned} n &\leqq |A|+|B|+|A'|+|B'|+\Big|\Big\{\bigcup_{\substack{j\notin C_1\cup C_2\\ j\neq 1,2}} N_j\cup\{x_0\}\Big\}\Big|+\Big|\Big\{\bigcup_{\substack{j\in C_1\cup C_2\\ j\neq 1,2}} N_j\Big\}\Big| \\ &\leqq 2f(c_1-1)+2f(c_2-1)+f(k-2)+\sum_{\substack{j\notin C_1\cup C_2\\ j\neq 1,2}} f(c_j) \\ &\leqq 4f(c_1-1)+f(k-2)+(k-1-c_1)f(c_1). \end{aligned}$$

If $c_1=k-1$ then this implies $n\leqq 5f(k-2)$, as desired.
If $c_1\leqq k-2$ then this implies (by (2))

$$\begin{aligned} n &\leqq 4\cdot 2^{-(k-2-c_1+1)}f(k-2)+f(k-2)+(k-1-c_1)2^{-(k-2-c_1)}f(k-2) = \\ &= (2^{c_1-k+3}+(k-1-c_1)2^{-(k-2-c_1)}+1)f(k-2) \leqq \\ &\leqq (2^{-2+3}+1+1)f(k-2) = 4f(k-2). \end{aligned}$$

Hence, in all cases

$$f(k)\leqq 5f(k-2),\quad k\geqq 3.$$

This proves (1′) and the proof of Theorem 1 is complete. ▮▮

We point out here that by a similar analysis, the colorings which achieve equality in Theorem 1 can be described completely, as follows:

If G_0 and G_1 are (disjoint) edge-colored complete graphs, define $[G_0, G_1](a)$ to be the complete graph having $V(G_0)\cup V(G_1)$ as its vertex set and with each edge having its original color if it is in G_0 or G_1 and having color a if it joins G_0 and G_1.

Similarly, if G_i, $0\leqq i\leqq 4$, are (disjoint) edge-colored complete graphs, define $[G_0, G_1, G_2, G_3, G_4](b,c)$ to be the complete graph having $\bigcup_{i=0}^{4} V(G_i)$ as its vertex set and with the edge e having its original color if it is contained in G_i for some i, and otherwise, having color b if it joins G_i to G_j, $i-j\equiv\pm 1 \pmod 5$ and having color c if it joins G_i to G_j, $i-j\equiv\pm 2 \pmod 5$.

The graphs for which equality holds in Theorem 1 are exactly those graphs on $f(k)$ vertices with k colors which can be formed by recursively applying the two preceding procedures where *new* colors are always used when combining graphs. Note that for k even, the $[G_0, G_1](a)$ construction is never used.

4. The case $s=4$

Theorem 2.

(4) $$f(4, k) = k+2 \quad \textit{for} \quad k \geqq 4.$$

Proof. To show that $f(4, k) \geqq k+2$ for $k \geqq 4$, consider the k-coloring χ of K_{k+2} with vertex set $[k+2]$ defined as follows:

$$\chi(1, 2) = \chi(3, 4) = 1, \quad \chi(1, 3) = \chi(2, 4) = 2, \quad \chi(1, 4) = \chi(2, 3) = 3,$$

$$\chi(1, i) = \chi(2, i) = 2 \quad \text{for} \quad 5 \leqq i \leqq k+2$$

$$\chi(3, i) = \chi(4, i) = 3 \quad \text{for} \quad 5 \leqq i \leqq k+2$$

$$\chi(i, j) = \min\{i, j\} - 1 \quad \text{for} \quad 5 \leqq i, j \leqq k+2$$

It is straightforward to check that in this coloring every K_4 spans exactly three colors.

The theorem will be proved if we can show

(5) $$f(4, k) \leqq k+2 \quad \text{for} \quad k \geqq 4.$$

Assume now that $\chi\colon E(K_n) \to [k]$ so that $c(K_4)=3$ for every $K_4 \subseteq K_n$. There are several cases.

Case 1. Suppose $c(K_3)<3$ *for every* $K_3 \subseteq K_n$.

In this case we will show by induction on k that $n \leqq k+1$, for $k \geqq 3$. Choose an arbitrary fixed vertex $x_0 \in V = V(K_n)$. As before define

$$N_i := \{v \in V\colon \chi(x_0, v) = i\}, \quad n_i := |N_i|, \quad C_i = C(N_i), \quad c_i = |C_i|,$$

where we can assume without loss of generality that $n_1 = \max_i \{n_i\}$.

Subcase 1. $n_i = 0$ for $i \neq 1$.

If $1 \in C_1$, say $\chi(\alpha, \beta) = 1$, $\alpha, \beta \in N_1$, then since $K_4(\alpha, \beta, \gamma, x_0)$ spans three colors for some $\gamma \in N_1$ then $K_3(\alpha, \beta, \gamma)$ must span three colors as well, contradicting the assumption of Case 1.

Thus, we can assume $1 \notin C_1$. Therefore, by induction

$$n_1 = |N_1| \leqq 1 + (k-1) = k,$$

and

$$n = 1 + n_1 \leqq k+1.$$

Of course, to complete this part of the argument, we must know that the induction gets started correctly at the beginning, for example, that if K_n is 3-colored so that every $K_4 \subseteq K_n$ spans three colors and no $K_3 \subseteq K_n$ spans three colors then $n \leqq 4$. This, in fact, is not difficult to show and its straightforward argument will not be given here.

Subcase 2. $n_i > 0$ for some $i \neq 1$.

We can assume (by relabelling if necessary) that $n_2 > 0$. By the hypothesis of Case 1, all edges in $[N_i, N_j]$ have colors i or j.

Suppose for some $i, j, i \neq j$, that $i \in C_j$, say $\chi(\alpha, \beta) = i$, $\alpha, \beta \in N_j$. This implies that for $\gamma \in N_i$, $K_4(x_0, \alpha, \beta, \gamma)$ has only colors i and j, which is a contradiction.

Thus, we can assume that if $i \neq j$ then $i \notin C_j$. As before if $i \in C_i$, say $\chi(\gamma, \delta) = i$ where $\gamma, \delta \in N_i$, then for $\alpha \in N_j$ we would have $K_4(x_0, \alpha, \gamma, \delta)$ with only colors i and j, which is impossible.

Claim. $C_i \cap C_j = \emptyset$ for $i \neq j$.

To see this, suppose $m \in C_i \cap C_j$, say $\alpha, \beta \in N_i$, $\gamma, \delta \in N_j$ with $\chi(\alpha, \beta) = m = \chi(\gamma, \delta)$. Now, if both i and j occur in $[\{\alpha, \beta\}, \{\gamma, \delta\}]$ then we would have a 3-colored K_3, which we are assuming does not occur. On the other hand, if at most one of i and j occur then $K_4(\alpha, \beta, \gamma, \delta)$ only has two colors, which is a contradiction. This proves the claim. ▮

Therefore,

$$n = 1 + \sum_i |N_i| = 1 + \sum_{c_i = 1} n_i + \sum_{c_i = 2} n_i + \sum_{c_i \geqq 3} n_i,$$

By induction,

$$c_i \geqq 3 \Rightarrow n_i \leqq 1 + c_i,$$

Also, if $c_i = 2$ then $n_i \leqq 3$ since otherwise we would have a K_4 in N_i with two colors. If $c_i = 1$ then $n_i \leqq 2$ since otherwise $\alpha, \beta, \gamma \in N_i$ results in $K(x_0, \alpha, \beta, \gamma)$ having only two colors. Hence, in every case, $n_i \leqq 1 + c_i$. Therefore

$$n \leqq 1 + \sum_{n_i > 0} (1 + c_i) = 1 + |\{i: n_i > 0\}| + \sum_{n_i > 0} c_i = 1 + k \tag{6}$$

by the preceding remarks. This completes the proof in Case 1.

Note that if some $K_3 \subseteq K_n$ has only *one* color then some other $K_3 \subseteq K_n$ must in fact have *three* colors. Thus, the only remaining case is:

Case 2. Some $K_3 \subseteq K_n$ has $c(K_3) = 3$. For definiteness, suppose $\alpha, \beta, \gamma \in V$ with $\chi(\beta, \gamma) = 1$, $\chi(\alpha, \gamma) = 2$, $\chi(\alpha, \beta) = 3$.

The argument will proceed by a sequence of claims.

Claim 1. All edges incident to $K_3(\alpha, \beta, \gamma)$ have colors 1, 2 or 3.

If not, we would have a K_4 with four colors. ▮

Claim 2. Any edge spanned by $V - \{\alpha, \beta, \gamma\}$ with color 1 must have a vertex v such that *all* edges incident to v have colors 1, 2 or 3.

Proof. Suppose $\lambda, \mu \in V - \{\alpha, \beta, \gamma\}$ with $\chi(\lambda, \mu) = 1$. Since $K_4(\beta, \gamma, \lambda, \mu)$ must span three colors and the only colors incident to β and γ are 1, 2 and 3 then *both* colors 2 and 3 must occur between $\{\beta, \gamma\}$ and $\{\lambda, \mu\}$. Thus, at least one of λ and μ forms a K_3 with β and/or γ having *three* colors. By Claim 1, this vertex can only be incident to edges with colors 1, 2 or 3. ▮

Let S denote the subset of vertices in V defined by:

$$S := \{v \in V: \chi(v, x) \in \{1, 2, 3\} \text{ for all } x \in V - \{v\}\}.$$

In other words, S consists of all vertices which are incident to only colors 1, 2 and 3. It follows from Claim 2 that $V-S$ spans no edge of color 1, 2 or 3 (since $\alpha, \beta, \gamma \in S$).

Claim 3. If $|V-S| \geqq 4$ then every K_3 in $V-S$ spans exactly two colors.

Proof. Suppose $K_3(\pi, \sigma, \delta) \subseteqq S' := V-S$ spans three colors, say a, b, c. By Claim 2, we must have $\{a, b, c\} \cap \{1, 2, 3\} = 0$. But also all edges in $[\{\alpha, \beta, \gamma\}, \{\pi, \sigma, \delta\}]$ must have colors a, b or c (since this is true for any edge incident to $K_3(\pi, \sigma, \delta)$). This now contradicts Claim 1.

On the other hand, suppose $K_3(\pi, \sigma, \delta) \subseteqq S'$ spans a single color a. Since $|S'| \geqq 4$, some $K_3 \subseteqq K_4(\pi, \sigma, \delta, \tau)$, $\tau \in S'$, spans three colors and we are back to the preceding case. This proves Claim 3. ∎

Thus, in S', all K_4's span *three* colors and all K_3's span *two* colors. Therefore, by (6)

$$|S'| \leqq 1 + c(S') \leqq 1 + k - 3 = k - 2. \tag{7}$$

Since we are assuming $k \geqq 4$ then we can also assume $|S'| \geqq 2$ (because S is incident to edges of only colors 1, 2 and 3).

Subcase 1. $|S'| \geqq 4$.

Claim 4. For each vertex $v \in S$, all edges $\{v, s'\}$, $s' \in S'$, have the *same* color.

Proof. Suppose the contrary, say, there exist $\alpha, \beta \in S'$ with $\chi(v, \alpha) = 1$, $\chi(v, \beta) = 2$. Choose $\gamma \in S'$ and consider the triangle $K_3(\alpha, \beta, \gamma)$. By Claim 3, $c(K_3(\alpha, \beta, \gamma)) = 2$ and these two colors cannot be 1, 2 or 3 (by Claim 2). Thus, $K_4(v, \alpha, \beta, \gamma)$ spans *four* colors, which is a contradiction. ∎

We next partition S into the sets M_i, $i = 1, 2, 3$ as follows:

$$M_i := \{u \in S \colon x \in S' \Rightarrow \chi(u, x) = i\}.$$

Claim 5. For $u \in M_i$, $v \in M_j$, $i \neq j$, we have $\chi(u, v) = i$ or j.

Proof. Suppose not, i.e., $\chi(u, v) = m \neq i, j$. For $\pi, \sigma \in S'$, we now have $c(K_4(u, v, \pi, \sigma)) = 4$, which is impossible. ∎

Claim 6. $i \notin C(M_i)$.

Proof. Suppose the contrary, say, $\chi(u, v) = i$, $u, v \in M_i$. Thus, for $\pi, \sigma \in S'$, $c(K_i(u, v, \pi, \sigma)) = 2$ a contradiction. ∎

Thus, we conclude that each M_i spans at most two colors and consequently, $|M_i| \leqq 3$.

Claim 7. $\sum_i |M_i| \leqq 4$.

Proof. Suppose $\sum_i |M_i| \geqq 5$. We can assume without loss of generality that $|M_1| \geqq 2$ and $|M_1| \geqq |M_2| \geqq |M_3|$.

Subcase (a). $|M_1|=3$. Then $|M_2|\geqq 1$ and M_1 must span both colors 2 and 3 (since otherwise three points in M_1 and one point in S' form a K_4 spanning only two colors). Suppose $u, v\in M_1$ with $\chi(u, v)=2$. By Claim 5, if $y\in M_2$, $\pi\in S'$, then $K_4(u, v, y, \pi)$ spans only colors 1 and 2, a contradiction.

The same argument applies to M_3 if $|M_3|\geqq 1$. Thus, we can assume $|M_2|=|M_3|=0$ and consequently, $\sum_i |M_i|=3$, contradicting the initial hypothesis in this claim.

Subcase (b). $|M_1|=2$. Let $M_1=\{u, v\}$. If $\chi(u, v)=2$ then as before, we must have $|M_2|=0$ so that $\sum_i |M_i|\leqq 4$, which again is a contradiction.

This proves the claim. ∎

Therefore, we can conclude that $\sum_i |M_i|=|S|\leqq 4$, and consequently, by (6)

$$n = |S|+|S'| \leqq k+2.$$

This completes the proof for Subcase 1.

We remark that $|S|=4$ can occur only when $|M_1|=|M_2|=2$, $|M_3|=0$ (or, of course, in general, where two of the $|M_i|$ are 2 and the other is 0).

Subcase 2. $|S'|=2$.

If for every $u\in S$ the colors of all $\{u, v\}$, $v\in S'$, are the same then we can apply the same arguments as in Subcase 1 and again conclude that $n\leqq k+2$ as desired. (The requirement that $|S'|\geqq 3$ was only needed in the proof of Claim 4.) Thus, we can assume without loss of generality that there are $x\in S$; $y, z\in S'$ such that $\chi(x, y)=1$ and $\chi(x, z)=2$. Note that all edges $\{x, w\}$ must have colors 1 or 2 (i.e., not 3) since otherwise $K_4(x, y, z, w)$ would have four colors.

Claim 8. No edge in $S-\{x\}$ has colors 1 or 2.

For, if $u, v\in S-\{x\}$ and $\chi(u, v)=1$ (or 2) then $K_4(x, u, v, y)$ spans only two colors. ∎

Therefore, all edges in $S-\{x\}$ have color 3. This implies $|S-\{x\}|\leqq 3$, i.e. $|S|\leqq 4$ and so, $n=|S|+|S'|\leqq 6\leqq k+2$.

This completes the treatment of Subcase 2 and Theorem 2 is proved. ∎ ∎

We mention here that by tracing through the preceding arguments carefully, it follows that the k-coloring of K_n given for the lower bound at the beginning of the proof of Theorem 2 is the *only* way (up to isomorphism) that $n=k+2$ can be achieved.

We also point out that by using arguments similar to those above, one can show

$$f(4, 3) = 9, \tag{8}$$

thereby filling in the missing value in Theorem 2 (and showing that (4) does not apply for $k=3$).

5. Concluding remarks

A natural question to raise at this point is for the values of $f(s, k)$ for $s \geqq 5$. Indeed, these values can be determined exactly, by arguments similar to (but much more complicated than) those given for the preceding results. Space limitations prevent us from giving more than just the statement:

Theorem 3.

$$f(s, k) = k+1 \quad \textit{for} \quad 5 \leqq s \leqq k. \tag{9}$$

On the other hand, when s is allowed to be larger than k then $f(s, k)$ increases dramatically, as the following result shows.

Theorem 4.

$$\big(1+o(1)\big)k^2 \leqq f(k+1, k) \leqq k^2+k. \tag{10}$$

The lower bound in (10) is relatively straightforward and follows from the stronger result

$$k^2 \leqq f(k+1, k) \tag{10'}$$

which holds for infinitely many values of k (e.g., k prime). The upper bound in (10) is considerably more difficult and will appear in a future paper. Conceivably, $f(k+1, k) \leqq k^2$ could hold for all $k \geqq 3$!

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